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Spectral Method Solutions for Some Laminar Channel Flows with Separation

A. Plotkin*
Naval Surface Weapons Center, White Oak, Maryland

Spectral method solutions to the plane slender-channel equations are investigated for some laminar, incompressible, high Reynolds number internal flows with separation. The equations are formulated in fourth-order stream-function form with the stream function taken as a Fourier series expansion plus a cubic polynomial in the transverse coordinate. Solutions are obtained for a sudden channel expansion, base in a channel, channel entry flow, and slowly diverging channel which agree well with the results of previous investigations.

Introduction

THERE is much current research activity devoted to the problem of analytic or numerical modeling of laminar high Reynolds number flow with separation. For many flows of aerodynamic interest, the boundary-layer equations of Prandtl provide a good first approximation of the Navier-Stokes equations when separation is absent. The standard form of the boundary-layer equations (pressure specified) for external flows is singular at separation (Goldstein,[1] Stewartson[2]) and can be integrated through separation only if an interaction between the boundary layer and external stream is allowed to occur. A description of a number of such interacting flow solutions is given by Davis and Rubin.[3]

Solutions to steady, two-dimensional, laminar, incompressible internal or channel flows that include separation appeared in the literature as early as 1910 (Blasius[4]) in the form of a Blasius series solution to the Navier-Stokes equations. For large values of Reynolds number, the slender-channel approximation (Williams[5]) is valid for a wide variety of channel flows. The resulting slender-channel equations are identical in form to the boundary-layer equations (with different scaling) but apply to a fully viscous flow. Numerical solutions of the slender-channel equations without separation are given by Bodoia and Osterle[6] for entry flow in a channel, by Paris and Whitaker[7] for the flow downstream of a plate in a channel, and by Blottner[8] for a variety of straight and curved channels.

Since the pressure is an unknown for the channel flows under consideration, it would appear possible to integrate the slender-channel equations through separation without the necessity of providing for the interaction that is required for external flows. For flows through slowly varying channels with regular separation (Brown and Stewartson[9]), Lucas[10] obtained a computer-extended Blasius series solution for a variety of channel shapes, and Eagles and Smith[11] calculated a finite-difference solution for a slowly diverging channel.

Kumar and Yajnik[12] solved the slender-channel equations in stream-function form for a number of separated channel flows including a sudden expansion. They departed from most previous investigators both in their choice of these parabolic equations for modeling flows with extensive recirculation zones and in the choice of a semianalytic spectral or integral relations method for the solution. The stream function is expanded in a series in the far wake eigenfunctions. Integration across the channel reduces the problem to integration of a coupled set of quasilinear, first-order, ordinary differential equations. With the use of at most five terms in the series, results are obtained that agree well with comparable numerical solutions of the Navier-Stokes equations. Plotkin[13] used a similar spectral method approach to calculate the development of the wake flowfield behind a symmetric cascade of finite-thickness flat plates. For this geometry, the eigenfunctions are Fourier series.

The spectral or integral relations methods are well-documented techniques for the solution of the laminar incompressible boundary-layer equations for attached flows.[14,15] No evidence could be found in the literature of failure of these techniques (for attached flow) due to the presence of a singularity in the resulting matrix equations. It is noted that in the studies by Kumar and Yajnik[12] (see Kumar[16]) and Plotkin,[13] singularities occurred that limited successful solutions to those flows with "moderate" recirculation zones and "limited" detail.

The goals of the current research are twofold. One is to explore further the applicability of the slender-channel equations to the proper modeling of separated flows with the hope of providing an explanation of the foregoing singular behavior. The other is to test the suitability of using Fourier series as universal expansion functions in the spectral method solution of the slender-channel equations. Gottlieb and Orszag[14] suggested Chebyshev series, and many investigators, such as Kumar and Yajnik,[12] used problem-dependent eigenfunction expansions.

In this paper, a spectral method using Fourier series expansions is used to solve the slender-channel equations for a sudden channel expansion, a base in a channel, channel entry flow, and flow in a slowly diverging channel.

Problem Formulation and Method of Solution

Consider the steady, two-dimensional, laminar incompressible flow in the parallel wall channels (sudden expansion, base in channel) shown in Fig. 1. The Cartesian coordinates x and y are nondimensionalized by the channel half-width, and velocities are nondimensionalized by the average downstream channel speed. The Navier-Stokes equations in nondimensional stream-function form are

$$\Psi_y \nabla^2 \Psi_x - \Psi_x \nabla^2 \Psi_y = R^{-1} \nabla^4 \Psi \tag{1}$$

where Ψ is the stream function, R is the Reynolds number, and ∇^2 and ∇^4 are the Laplacian and biharmonic operators. Kumar and Yajnik[12] stated that, for large values of the Reynolds number, except in the immediate neighborhood of the origin, the streamwise length scale is of $\mathcal{O}(R)$, the transverse length scale is of $\mathcal{O}(1)$, the streamwise velocity component is of $\mathcal{O}(1)$, the transverse velocity component is of $\mathcal{O}(R^{-1})$, and the pressure is of $\mathcal{O}(1)$. If the contracted

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*Consultant, Applied Mathematics Branch; also Professor, Department of Aerospace Engineering, University of Maryland, College Park. Associate Fellow AIAA.

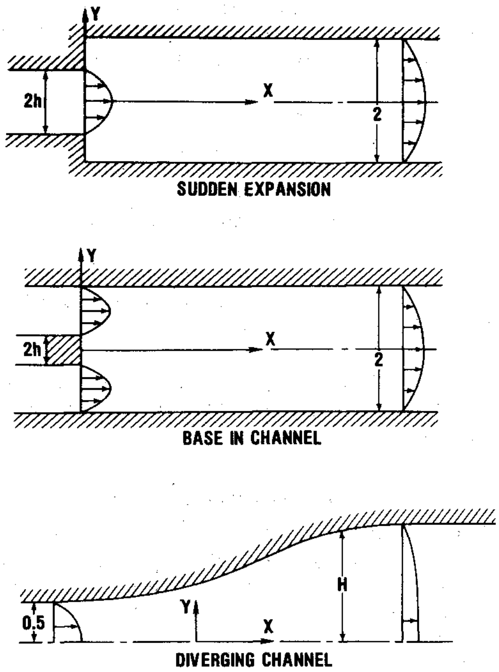

Fig. 1 Flow configurations and coordinate system.

streamwise coordinate $X=x/R$ is used, Eq. (1) becomes

$$\Psi_{yyyy}=\Psi_y\Psi_{yyX}-\Psi_X\Psi_{yyy}+R^{-2}$$
$$\times[\Psi_y\Psi_{XXX}-\Psi_X\Psi_{yXX}-2\Psi_{yyXX}]-R^{-4}\Psi_{XXXX} \qquad (2)$$

and the large Reynolds number limit of Eq. (2) yields the stream-function form of the slender-channel equations[5]

$$\Psi_{yyyy}=\Psi_y\Psi_{yyX}-\Psi_X\Psi_{yyy} \qquad (3)$$

Note that this is the Prandtl boundary-layer equation (Van Dyke[17]) with different scaling. [The standard boundary-layer equation has a streamwise length scale of $\mathcal{O}(1)$ and a transverse length scale of $\mathcal{O}(R^{-1/2})$.] The standard third-order form of the boundary-layer equation is recovered by an integration of Eq. (3) with respect to y and an identification of the streamwise pressure gradient as the constant (function of X only) of integration. The pressure is an unknown for these internal flows and will be determined with the use of an integral momentum equation.

On the channel walls, the stream function must be constant and the velocity must vanish. The wall boundary conditions are

$$\Psi(X,\pm 1)=\pm 1 \qquad (4a)$$

$$\Psi_y(X,\pm 1)=0 \qquad (4b)$$

Note that Eq. (3) is first order in X whereas the original Navier-Stokes equation (1) is fourth order. Therefore, only one boundary condition can be applied at $X=0$, and in general this would come from matching with a local or inner solution valid in the neighborhood of the origin. In the contracted coordinate X, the streamwise extent of the region of validity of the local solution vanishes. The initial condition used here is

$$\Psi(0,y)=\Psi_0(y) \qquad (5)$$

which is equivalent to prescribing the initial streamwise velocity profile. The initial streamline slopes cannot be prescribed.

A perturbation stream function ψ is introduced which satisfies the following mathematical problem:

$$\Psi=(3/2)(y-y^3/3)+\psi \qquad (6a)$$

$$\psi_{yyyy}-(3/2)(1-y^2)\psi_{yyX}-3\psi_X=\psi_y\psi_{yyX}-\psi_X\psi_{yyy} \qquad (6b)$$

$$\psi(X,\pm 1)=\psi_y(X,\pm 1)=0 \qquad (6c)$$

$$\psi(0,y)=\Psi_0(y)-(3/2)(y-y^3/3) \qquad (6d)$$

Kumar and Yajnik[12] solved the foregoing problem with an eigenfunction expansion method that is a variation of integral relations. The expansion functions for their method must be determined numerically which appears to present a serious obstacle to the calculation of more than five terms in the expansion and the subsequent establishment of more positive evidence of convergence. The method is also a spectral method,[14] with the streamwise coordinate as the time-like variable. For problems such as the above with nonperiodic boundary conditions, Orszag[14] suggested the use of Chebyshev polynomial expansions. It is the intention of the author to study the suitability of following an alternative suggestion of Orszag to use Fourier series plus a low-order polynomial to satisfy the boundary conditions. This choice leads to a less tedious generation of the final form of the governing equations.

It can be easily seen that the following expansion satisfies the boundary conditions of Eq. (6c) identically:

$$\psi(X,y)=\sum_1^N a_n(X)\sin n\pi y+\frac{y-y^3}{2}\sum_1^N n\pi a_n(X)\cos n\pi \qquad (7)$$

The Fourier series is truncated after N terms. Equation (7) is substituted into Eq. (6b) and, after the result is multiplied by $\sin m\pi y$ and integrated across the channel, the following set of coupled quasilinear first-order ordinary differential equations is obtained:

$$\sum_n\Big[B_{mn}-\sum_q C_{mnq}a_q\Big]a_n'=-m^4\pi^4 a_m \qquad m=1,N \qquad (8)$$

where

$$B_{mn}=\frac{3}{2}n^2\pi^2\int_{-1}^{1}(1-y^2)\sin n\pi y\sin m\pi y\,dy$$
$$+3n\pi\cos n\pi\int_{-1}^{1}(y-y^3)\sin m\pi y\,dy-3\delta_{mn} \qquad (9a)$$

and

$$C_{mnq}=\pi^3(q^3-qn^2)\int_{-1}^{1}\sin n\pi y\sin m\pi y\cos q\pi y\,dy$$
$$-\cos n\pi\int_{-1}^{1}\Big(3\pi^2 nqy+\pi^4 nq^3\frac{y^3-y}{2}\Big)\cos q\pi y\sin m\pi y\,dy$$
$$+\cos q\pi\int_{-1}^{1}\Big(3q\pi+\pi^3 qn^2\frac{3y^2-1}{2}\Big)\sin n\pi y\sin m\pi y\,dy$$
$$+3\pi^2 nq\cos n\pi\cos q\pi\int_{-1}^{1}y^3\sin m\pi y\,dy \qquad (9b)$$

where the prime denotes differentiation with respect to X and δ_{mn} is the Kronecker delta. To obtain the initial values of the functions $a_m(X)$, substitute Eq. (7) into the initial condition in Eq. (6d), multiply by $\sin m\pi y$, and integrate across the channel. The resulting coupled set of linear equations is

$$\sum_n A_{mn} a_n(0) = b_m \qquad m=1,N \tag{10}$$

where

$$A_{mn} = \delta_{mn} - 6n\cos m\pi \cos n\pi / \pi^2 m^3 \tag{11a}$$

and

$$b_m = \int_{-1}^{1} \Psi_0(y) \sin m\pi y \, dy + \cos m\pi \left(\frac{2}{m\pi} + \frac{6}{\pi^3 m^3} \right) \tag{11b}$$

Strictly speaking, the problem represented by the ordinary differential equation set (8) with the initial conditions of Eq. (10) is the development of the channel flow downstream of an initial station with a given streamwise velocity profile. The flow configuration upstream of this initial station enters the problem only through the initial conditions. Therefore the choice of an appropriate initial profile is critical to the modeling. It is assumed that the flow far upstream of the origin is fully developed. Kumar and Yajnik[12] and Plotkin[13] showed that parabolic initial profiles are suitable for the sudden expansion and base in a channel flow, and they will be assumed for our solution.

For the sudden expansion, h is the ratio of upstream to downstream channel heights, and the initial profile is taken to be

$$\begin{aligned} \Psi_0 &= 1 \quad h \le y \le 1 \\ &= \frac{3}{2} [y/h - (y/h)^3/3] \qquad |y| \le h \\ &= -1 \quad -1 \le y \le -h \end{aligned} \tag{12a}$$

and

$$b_m = \frac{6}{\pi^3 m^3} \left(\cos m\pi - \frac{\cos m\pi h}{h^2} + \frac{\sin m\pi h}{m\pi h^3} \right) \tag{12b}$$

For the base in a channel, h is the ratio of base height to channel height, and the initial profile is taken to be

$$\begin{aligned} \Psi_0 &= 0 \qquad |y| \le h \\ &= 1 - [3(y-h)(1-y)^2 + (1-y)^3]/(1-h)^3 \\ &\qquad h \le |y| \le 1 \end{aligned} \tag{13a}$$

and

$$b_m = \frac{-12}{\pi^3 m^3 (1-h)^2} \times \left[\cos m\pi + \cos m\pi h + \frac{2\sin m\pi h}{m\pi(1-h)} + \frac{6\cos m\pi}{\pi^3 m^3} \right] \tag{13b}$$

For the problem of entry flow in a channel, the channel height is constant and the initial profile is uniform. Therefore,

$$\Psi_0 = y \qquad -1 \le y \le 1 \tag{14a}$$

and

$$b_m = 6\cos m\pi / \pi^3 m^3 \tag{14b}$$

Once the stream function is found from the solution of Eq. (8), the pressure field can be obtained as follows. Equation (3) can be integrated once with respect to y to yield

$$\Psi_{yyy} + \Psi_X \Psi_{yy} - \Psi_y \Psi_{yX} = p'(X) \tag{15}$$

where the pressure p has been made nondimensional by the product of the density and the square of the characteristic speed. Equation (15) is integrated from $y=0$ to 1 to yield

$$\frac{d}{dX} \int_0^1 (p + \Psi_y^2) dy = \Psi_{yy}(X,1) \equiv -\Omega_w(X) \tag{16}$$

where Ω_w is the nondimensional wall vorticity. Equation (16) is an integral momentum equation for the streamwise gradient of the channel mean pressure.

Slowly Diverging Channel

The sudden expansion and the base in a channel flow are both flows where the geometry fixes the separation point at a corner and where the calculation begins inside a recirculation zone. These examples do not allow for the study of regular separation along a smooth wall including the prediction of the location of the separation point. An ideal example for the study of regular separation using the slender-channel equations was given by Eagles and Smith.[11] They considered the slowly diverging channel shown at the bottom of Fig. 1. The channel walls are described by

$$y = \pm(1 + \tfrac{1}{2}\tanh\lambda X) = \pm H(X) \tag{17}$$

The walls are essentially parallel far upstream and downstream, and λ is a measure of the steepness of the wall slope. The flow is fully developed far upstream in the channel, and the lengths and velocities are made nondimensional by scaling with the upstream channel height and one-half the mean speed. With the use of the scaled transverse coordinate $\eta = y/H$, the mathematical problem for the stream function that corresponds to Eqs. (3-5) is found to be

$$\Psi_{\eta\eta\eta\eta} + 2H'\Psi_\eta \Psi_{\eta\eta} + H(\Psi_X \Psi_{\eta\eta\eta} - \Psi_\eta \Psi_{\eta\eta X}) = 0 \tag{18a}$$

$$\Psi(X, \pm 1) = \pm 1 \tag{18b}$$

$$\Psi_\eta(X, \pm 1) = 0 \tag{18c}$$

$$\Psi(-\infty, \eta) = (3\eta - \eta^3)/2 \tag{18d}$$

A perturbation stream function is introduced,

$$\Psi = \psi + (3\eta - \eta^3)/2 \tag{19a}$$

that satisfies

$$\begin{aligned} &\psi_{\eta\eta\eta\eta} - H[3\psi_X + (3/2)(1-\eta^2)\psi_{\eta\eta X}] + H[\psi_X \psi_{\eta\eta\eta} - \psi_\eta \psi_{\eta\eta X}] \\ &= -2H'[\psi_\eta \psi_{\eta\eta} - 3\eta\psi_\eta + (3/2)(1-\eta^2)\psi_{\eta\eta} \\ &- (9\eta/2)(1-\eta^2)] \end{aligned} \tag{19b}$$

The series expansion corresponding to Eq. (7) is introduced,

$$\psi(X,\eta) = \sum_1^N a_n(X) \sin n\pi\eta + [(\eta - \eta^3)/2] \sum_1^N n\pi a_n \cos n\pi \tag{20}$$

and the ordinary differential equation set that corresponds to

Eq. (8) is found to be

$$\sum_n \left[B_{mn} - \sum_q C_{mnq} a_q\right] a_n' = -m^4 \pi^4 a_m H^{-1} - 2H' H^{-1} E_m \qquad m=1,N \tag{21}$$

where the one new coefficient, E_m, arises from an integration of the terms on the right-hand side of Eq. (19b). The initial conditions of Eq. (18d) are satisfied exactly by requiring that

$$a_m(-\infty) = 0 \qquad m=1,N \tag{22}$$

Results and Discussion

The channel centerline velocity is obtained from Eq. (7) as

$$u_c(X) = 1.5 + \sum_1^N n\pi a_n(X)\left[1 + \frac{1}{2}\cos n\pi\right] \tag{23}$$

The length of the recirculation zone X_R is determined from

$$\Omega_w(X_R) = 0 \tag{24}$$

where the wall vorticity function Ω_w is given in Eq. (16). For the slowly diverging channel,

$$u_c = H^{-1}\Psi_\eta \quad \text{and} \quad \Omega_w = -H^{-2}\Psi_{\eta\eta} \tag{25}$$

The ordinary differential equation set in Eq. (8) can be expressed as the matrix equation

$$\underset{\sim}{D}\underset{\sim}{a}' = -\underset{\sim}{\Lambda}\underset{\sim}{a} \tag{26}$$

where $\underset{\sim}{a}$ is a column matrix with elements $a_1, a_2, \ldots, a_N$ and $\underset{\sim}{D}$ and $\underset{\sim}{\Lambda}$ are $N \times N$ square matrices. At a given station X, the derivative $\underset{\sim}{a}'$ can be calculated if $\underset{\sim}{a}$ is known, provided the determinant det $\underset{\sim}{D}$ is nonzero. The matrix equations are integrated numerically using a fourth-order, variable-step-size Runge-Kutta-Fehlberg method.

Finite-difference solutions to the slender-channel equations have been shown to be good high Reynolds number approximations to the Navier-Stokes equations for the unseparated entry flow in a channel (see Van Dyke[18]) and flow downstream of a plate in a channel.[7] These examples can be used to test the ability of the Fourier series spectral method to solve the slender-channel equations.

In Fig. 2 the channel centerline velocity for the entry flow in a channel is given for the finite-difference results of Bodoia and Osterle,[6] the five-term solution of Kumar and Yajnik,[12] and the seven-term Fourier series solution. The agreement between the current results and the finite-difference solution of the partial differential equations is quite encouraging. It is interesting to note that the Fourier series results are superior to those obtained using eigenfunctions for this particular case.

In Fig. 3 the channel centerline velocity for the flow downstream of a plate in a channel (the merging of two Poiseuille profiles) is given for the finite-difference results of Paris and Whitaker[7] and the five-term Fourier series solution. This problem is the limiting case of a base in a channel with $h=0$. The agreement is good.

For the separated flow examples, the sudden channel expansion, and the base in a channel, Kumar and Yajnik[12] encountered singularities in the matrix equations which placed a limit on the number of terms that could be obtained for a particular geometric configuration. For example, for the sudden expansion, no more than two terms could be calculated in the eigenfunction expansion when $h<0.5$. When a calculation was started with $N \geq 3$, the determinant of the matrix equation system became zero during the calculation.

Since Kumar and Yajnik[12] presented results for at most five terms in their expansion, it is difficult to assess the convergence properties of their technique. They claimed good agreement between their results for the sudden expansion with $h=0.5$ and numerical solutions of the Navier-Stokes equations. However, Agarwal[19] reported that his Navier-Stokes computations agreed better with their three-term solution than with their five-term solution.

The present results exhibit the same qualitatively singular behavior as that experienced by Kumar and Yajnik.[12] The maximum number of terms that can be obtained in the Fourier series expansion increases as h approaches 1 for the sudden channel expansion and as h approaches zero for the base in a channel. For flows with "moderate" recirculation zones present, solutions with reasonable convergence properties (accuracy to three significant figures, for example) can be obtained with N_M terms, and no solution may exist for $N=N_M+1$. Similar behavior has been reported for finite-difference solutions of the slender-channel equations with separation. For example, Blottner[8] reported a stable numerical solution for a channel flow calculation for one

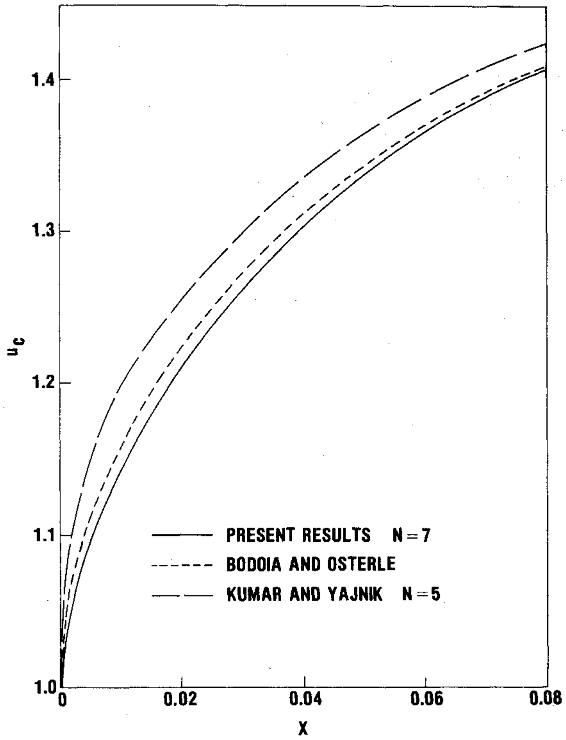

Fig. 2 Channel centerline velocity for entry flow in a channel.

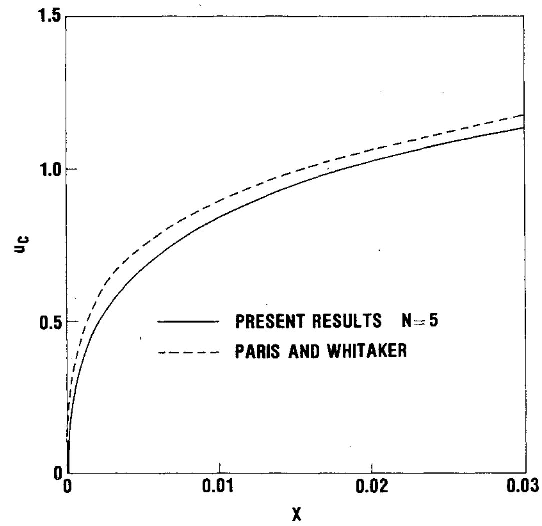

Fig. 3 Channel centerline velocity for plate in a channel.

value of the streamwise grid spacing that could not be repeated as the grid size was reduced.

Since the sudden channel expansion with $h=0.5$ has become a standard test case in the literature, results will be presented even though only three terms can be calculated. Figure 4 compares the present results for the channel centerline velocity with those of the five-term solution of Kumar and Yajnik.[12] The agreement is excellent. Figure 5 compares the present results for the wall vorticity with those of the three- and five-term solutions of Kumar and Yajnik.[12] The Fourier-series solution seems to follow the five-term result. For the sudden channel expansion with $h=0.75$, the five- and six-term Fourier-series solutions agree to four significant figures in the channel centerline velocity. This result is compared with that of the five-term Kumar and Yajnik expansion in Fig. 6, and the agreement is excellent.

The slowly diverging hyperbolic tangent channel [Eq. (17)] is an excellent example with which to study regular separation along a smooth wall including prediction of the location of the separation point. Eagles and Smith[11] obtained finite-difference solutions of the slender-channel equations using a forward-marching technique. For small values of λ, the flow in the channel is attached. As λ is increased, regular separation is first achieved on the channel wall for $\lambda \cong 15.5$ and flows with separation bubbles are calculated for $\lambda \leq 18$ with an accuracy in the wall vorticity of 10^{-4}.

Figure 7 compares the wall vorticity from the Fourier series computation with the results of Eagles and Smith.[11] The agreement is seen to be excellent for $\lambda=8$ and worsens as λ increases. The Fourier series spectral method predicts

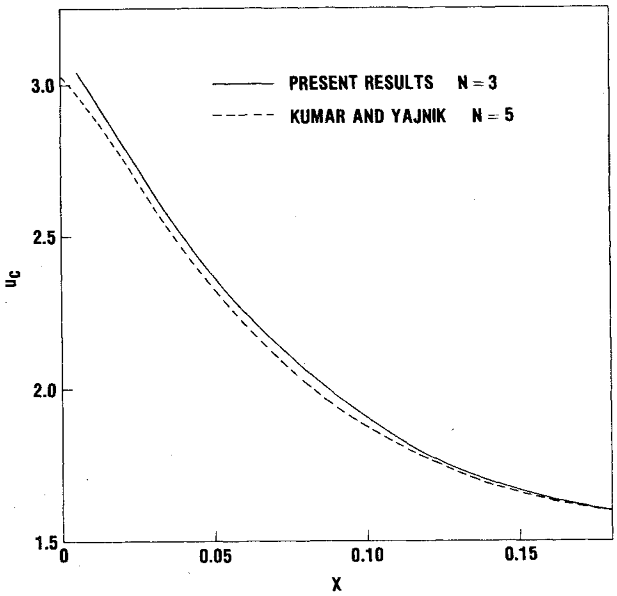

Fig. 4 Channel centerline velocity for sudden expansion: $h=0.5$.

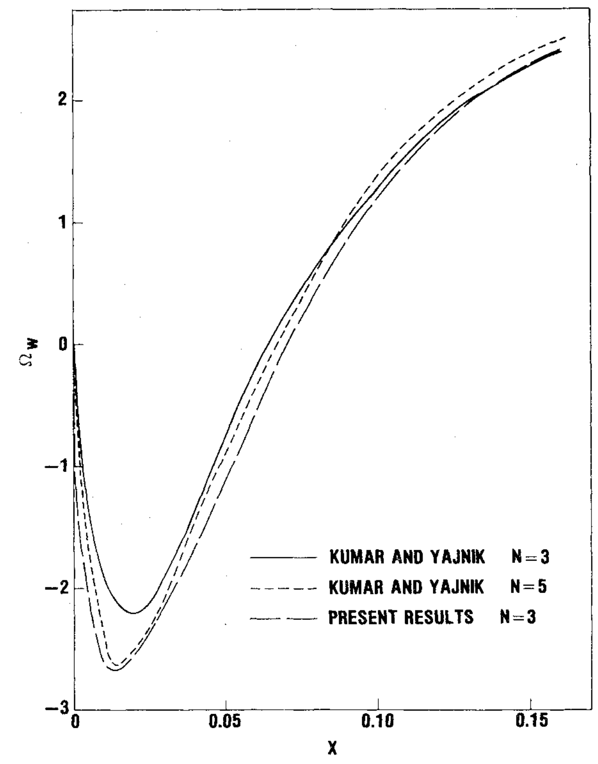

Fig. 5 Wall vorticity for sudden expansion: $h=0.5$.

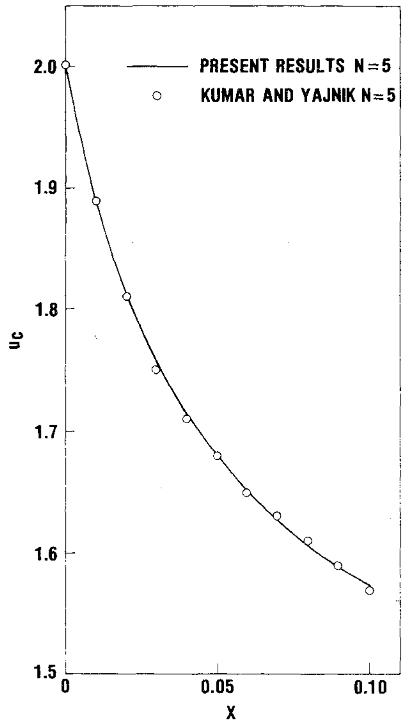

Fig. 6 Channel centerline velocity for sudden expansion: $h=0.75$.

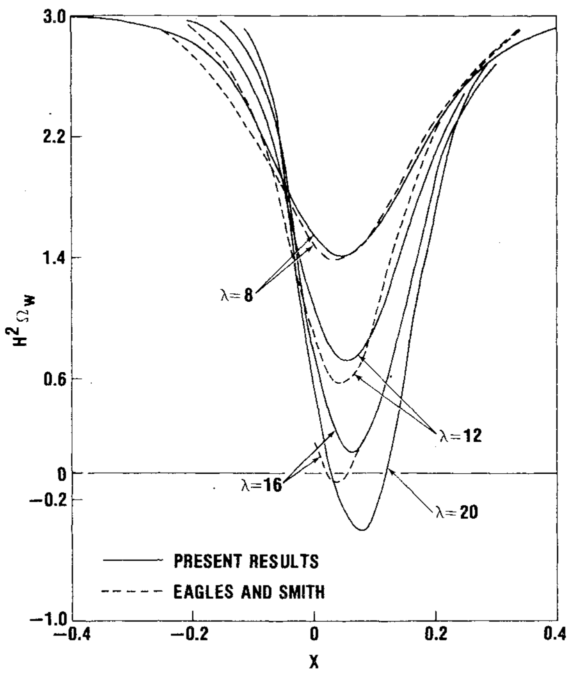

Fig. 7 Wall vorticity for slowly diverging channel.

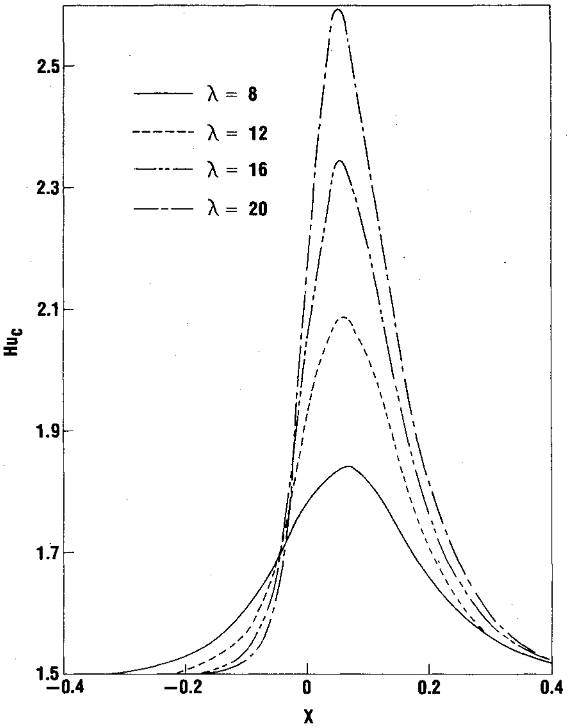

Fig. 8 Channel centerline velocity for slowly diverging channel.

separation to occur first for $\lambda \cong 17.0$. The convergence properties for the slowly diverging channel example are very good. Accuracy to three decimal places is achieved with three terms for $\lambda = 8$ and four terms for $\lambda = 16$. Once the flow separates, for $\lambda > 17$, a five-term solution cannot be calculated. For $N = 5$, the matrix equation becomes singular far upstream of the separation point. Lucas[10] calculated the same channel using computer-extended Blasius series, but did not obtain a converged solution. Figure 8 shows the channel centerline velocity as a function of λ for the Fourier series method. It is interesting to note the sharp increase in u_c in the region along the wall in the neighborhood of the separation bubble.

Conclusions

A Fourier series spectral method has been investigated for use in the solution of the slender-channel equations for steady, two-dimensional, laminar, incompressible flow in channels at large Reynolds number. This is a companion study to the research of Kumar and Yajnik,[12] who used an eigenfunction expansion spectral method, and Eagles and Smith,[11] who used finite differences to solve the slender-channel equations.

For problems with attached flow on the straight channel walls,[6,7] the Fourier series spectral method provides good agreement with corresponding finite-difference solutions of the partial differential equations. The solutions are seen to converge within a relatively small number of terms. For the channel entry flow, the Fourier series solution appears to be superior to that obtained using the eigenfunction expansion. For the slowly diverging channel problem, for values of λ where the flow is attached, the excellent agreement with the results of Eagles and Smith[11] for $\lambda = 8$ is seen to worsen with increasing λ. It is noted that the agreement for the channel centerline velocity is substantially better than that for the wall vorticity. A possible explanation for the discrepancy in wall vorticity lies with the assumed form of the stream function in the current method. Even though the velocity boundary conditions at the wall are satisfied exactly with the cubic polynomial, the vorticity near the wall may not be adequately modeled, especially near separation. Polynomial representation of wall boundary conditions for a related pseudospectral calculation was discussed by Roache.[20]

For the separated flow sudden channel expansion and base in a channel examples, the Fourier series method exhibits the same qualitatively singular behavior as that experienced by Kumar and Yajnik[12] and Plotkin.[13] For the sudden expansion with h near 1 and the base in a channel with h near zero, apparently convergent solutions with "moderate" recirculation zones present can be obtained with only a small number of terms in the expansion. Reasonable agreement has been shown between the results of Kumar and Yajnik[12] and some Navier-Stokes results of Agarwal,[19] and for the sudden expansion with $h = 0.75$, there is good agreement between the Fourier series and eigenfunction expansion results.

The singular behavior experienced by the spectral method matrix equation system is similar to that encountered when the slender-channel equations are integrated with the use of a finite-difference marching technique (see e.g., Ref. 8). The slender-channel equations are parabolic and when a marching technique is used, there is no mechanism for upstream influence in the solution. When reversed flow is present, however, information must propagate upstream, and the equations possess an elliptic character. It is postulated that it is the ill-posed nature of the problem when the foregoing techniques are used to march through regions of reversed flow that leads to the singular behavior.

The use of a Fourier series expansion plus a cubic polynomial in a spectral method solution of the slender-channel equations appears to be a reasonably promising approach and should be explored further in the attempt to solve related problems. Applicability of the slender-channel equations to proper modeling of separated flows, however, remains an open question. The relationship of the "converged" solutions presented here to the corresponding exact solutions remains to be resolved.

The slowly diverging channel problem would appear to be an excellent test case. Two solutions to the slender-channel equations are available: the finite-difference results of Eagles and Smith[11] and the Fourier series spectral method results presented here. What is lacking is an exact solution. Smith[11] stated that the slender-channel equations "provide a uniformly valid leading order description of the entire flow field" and that a finite-difference solution of these equations would provide the correct flowfield as long as the direction of integration followed the stream direction.[21] A technique for implementation of the procedure is given by Williams.[22] The final ingredient necessary for partial resolution of the foregoing question is an accurate numerical solution of the complete Navier-Stokes equations at large Reynolds number.

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